Grade 6 Math Circles

October 4/5/6, 2022

Number Systems

Introduction to Exponents

Mathematicians excel at finding ways to write less, especially by using symbols and abbreviations to represent words.

Multiplication is an operation that simplifies the repeated addition of a single number. For example, $4+4+4+4+4+4 = 4 \times 6$.

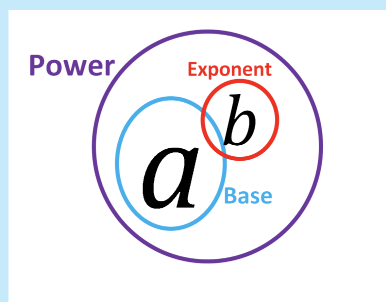
There is also an operation, called **exponentiation**, which simplifies the repeated multiplication of a single number. For example, $2 \times 2 \times 2 \times 2 \times 2 = 2^5$.

Note that **integers** are positive whole numbers, their negative counterparts, and 0.

In general, if a and b are integers, $a^b = \underbrace{a \times a \times \ldots \times a}_{b \text{ times}}$.

The entire expression is called the **power**, a is the **base**, and b is the **exponent**.

This expression is read as "a to the exponent b", which just means a multiplied by itself b times.



For any integer a, such that $a \neq 0$, the power a^0 is equal to 1.

The **powers** of an integer a are all possible powers with base a. So, the powers of 2 include $2^0, 2^1, 2^2, 2^3$, and so on.

Exercise 1

Rewrite each expression as a power.

a) $8 \times 8 \times 8 \times 8 \times 8 \times 8$ b) $1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1$ c) $5 \times 5 \times 5$

d) $7 \times 7 \times 7 \times 7 \times 7$ e) $10 \times 10 \times 10 \times 10$

Exercise 2

Rewrite each number as a power. Do not write a power with the exponent 1. There might be more than a single base possible for some numbers.

a) 49 b) 16 c) 1000 d) 1 e) 25

Exercise 3

Evaluate the following powers.

a) 2^3 b) 9^1 c) 6^2 d) 18^0 e) 10^6 f) 3^3

Number Systems and Bases

Just as there are different languages, there are different **number systems**.

"Mathematics", "Mathématiques", and "Matemática" are all words that mean mathematics in the English, French, and Spanish languages respectively. They are written differently but have the same meaning.

If number systems are languages, then numbers are words. And just like words in different languages, numbers in different number systems can look different but have the same meaning.

More formally, a **number system** is a system for writing numbers using digits and symbols.

A **digit** is any of the following: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Digits are the most common symbols that are used to form numbers.

The **base** of a number system is the number of symbols that are used to form numbers in the number system.

The **decimal number system** is the number system which is most used today. This is the number system that uses the ten digits and so this system is in base 10.

Stop and Think
Why is the most common number system, the decimal system, in base 10? Why not base 9 or base 11?

The decimal system is the number system that is used in our everyday lives. To form a number, the digits are ordered into positions which are named, from right to left: the ones, the tens, the hundreds, the thousands, the ten thousands, and so on. The values associated with the digits change based on their positions.

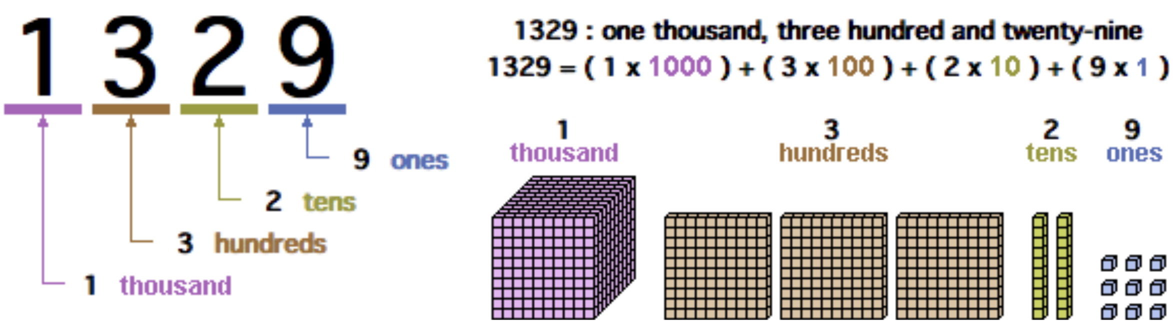


Figure 1: Retrieved from LearnAlberta.

The decimal system is just one of many number systems. In general, number systems can be organized into two categories: non-positional and positional number systems. In this lesson, we will focus on positional number systems.

The Importance of Number Systems

Learning about different number systems might seem unimportant. We already have the decimal system, why would we need any other?

It turns out that there are many real-world applications that use number systems. Most of these applications are connected to computers.

Computers only understand numbers, they can't process languages. In fact, computers can only understand 0 (off/no/false) and 1 (on/yes/true).

There are a few number systems that are widely used in computing: binary, octal, and hexadecimal. Binary is the base 2 number system with symbols 0 and 1. Thus, binary is the only system that computers can understand. Other numbers systems humans use in computing are converted into binary so that they can be understood by computers.

There are many specific applications linked to each number system. For example, the binary system can be used to represent the pixels in an image and can be used to store data. In general, the other number systems are used to interpret long binary numbers since larger numbers will require fewer symbols when working with larger bases. An interesting application of the hexadecimal system is to represent digital colours. For example, the hexadecimal number 000000 is black and the hexadecimal number FFFFFF is white.

Number systems are also used in the encryption of data. Converting information into another number system can make it difficult for someone else to understand.

Positional Number Systems

Positional number systems (sometimes called place value systems) are number systems that depend on the positions of symbols within the number. A number in a positional number system is represented as a string of symbols.

Recall that the base of a number system is the number of different symbols used in the system.

The decimal system, or base 10 system, is a positional number system. Using the exponentiation operation, we can expand a positive integer based on its digits. For example,

$$\begin{aligned} 49238 &= 40000 + 9000 + 200 + 30 + 8 \\ &= 4 \times 10^4 + 9 \times 10^3 + 2 \times 10^2 + 3 \times 10^1 + 8 \times 10^0 \end{aligned}$$

In the above expansion, each digit is multiplied by a power of 10, which increases from right to left. As a result, each position has a weight that is ten times more than the previous position, where the weight of the first position is 1.

This can be generalized to any positional number system with base b, where each position has a weight that is b times more than the previous position and the weight of the first position is 1.

Exercise 4

List the symbols used for each positional number system, in increasing order of their equivalent decimal system values.

a) Base 2 (binary) b) Base 8 (octal) c) Base 16 (hexadecimal)

Notation of Number Systems

Just like words that can look the same but have different meanings in different languages, there can be numbers that have the same digits or symbols but represent different values in different number systems.

(a) Une tour - French (b) A tour - English

Figure 2: (a) Retrieved from Amazon, (b) Retrieved from Viator.

Using notation that specifies the base of a number system can prevent a lot of confusion when working with multiple number systems. The common notation is to include the base of the number system as a subscript for the number.

Typically, if a number does not specify the base as a subscript, the number has base 10.

Example 1

- 101_2 is in base 2
- 101_{10} is in base 10
- 3850_9 is in base 9
- 3850 is in base 10

Exercise 5

In Exercise 4, a standard convention was introduced as to what symbols are used in certain number systems. Write the following numbers using the proper notation for three number systems to which they could belong.

a) 3729 b) 104102 c) $32EJ9$

Converting to Decimal

Positional number systems are basically different ways of writing the same numbers. Using the decimal system as a point of reference, we can define a structure for all positional number systems.

In every positional number system, a number is represented as a string of symbols. Most positional number systems will use digits and then letters as their symbols, but any symbols can be used. Every symbol has a position in the number; starting at 0 for the rightmost and increasing by 1 for each position we move to the left. Recall that the base of the system is a positive integer b. We can now define the **weight** of a position p to be the power b^p. Every symbol is given a value from 0 to $b-1$ (in decimal which is our point of reference). So, every number in a positional number system can be represented as an expansion similar to the example with 49238, where the expansion is the sum of the value of each digit multiplied by the weight of its position, and the result is a number in base 10.

$\square\square ... \square\square\square$ can be represented as $(\square \times b^{n-1}) + (\square \times b^{n-2}) + ... + (\square \times b^2) + (\square \times b^1) + (\square \times b^0)$ in decimal. Here $\square$ is a symbol, $\square\square ... \square\square\square$ is a string of symbols representing a number in a number system, b is the base of the number system, and n is the length of the number.

The above expansion is not only a structure for positional number systems, it is also the method by which we convert a number from any positional number system into the decimal number system. The representation above can be used to simplify a number in a positional number system to the equivalent decimal number.

Example 2

To convert 101_2 to decimal, we write out the expansion.

$$\begin{aligned} 101_2 &= (1_2 \times 2^2) + (0_2 \times 2^1) + (1_2 \times 2^0) \\ &= (1 \times 4) + (0 \times 2) + (1 \times 1) \\ &= 4 + 0 + 1 \\ &= 5 \end{aligned}$$

Thus, $101_2 = 5_{10}$.

Recall that numbers without subscript notation are in the decimal system.

Example 3

To convert 3850_9 to decimal, we write out the expansion.

$$\begin{aligned} 3850_9 &= (3_9 \times 9^3) + (8_9 \times 9^2) + (5_9 \times 9^1) + (0_9 \times 9^0) \\ &= (3 \times 729) + (8 \times 81) + (5 \times 9) + (0 \times 1) \\ &= 2187 + 648 + 45 + 0 \\ &= 2880 \end{aligned}$$

Thus, $3850_9 = 2880_{10}$.

Converting from Decimal

Converting a number from the decimal system to any positional number system with base b is a little more complicated. In order to make the conversions easier to handle, we'll be using a table like the following.

Base 10	b^4	b^3	b^2	b^1	b^0	Base b

First, find the largest power of b that is less than (or equal to) the decimal number. Subtract the power from the decimal number and put a tally in the box below the power. Then, find the largest power of b that is less than the new decimal number. Subtract the new power from the new decimal number and put a tally in the box below the power. Continue this pattern until the decimal number reaches zero. Convert the number of tallies in each box into the equivalent symbols in the base b number system and write that symbol in the box below. The last row of boxes from left to right is the new base b number. In all number systems, we do not write leading zeros.

Example 4

To convert 101_{10} to base 2, we will fill in the conversion table by writing a tally mark every time we subtract a power.

First, we note that $2^6 = 64$ is the largest power of 2 that is less than 101 and $101 - 64 = 37$. Then, $2^5 = 32$ is the largest power of 2 that is less than 37 and $37 - 32 = 5$. Then, $2^2 = 4$ is the largest power of 2 that is less than 5 and $5 - 4 = 1$. Then, $2^0 = 1$ is the largest power of 2 that is less than (or equal to) 1 and $1 - 1 = 0$.

Now, we fill in the bottom row by writing the numeric value of the tallies for each column. The number on the bottom row, read from left to right, is the base 2 number equivalent to the decimal number. So, 101_{10} in a base 2 system is 1100101_2.

<table>
<tr><th>Base 10</th><th>$2^7 = 128$</th><th>$2^6 = 64$</th><th>$2^5 = 32$</th><th>$2^4 = 16$</th><th>$2^3 = 8$</th><th>$2^2 = 4$</th><th>$2^1 = 2$</th><th>$2^0 = 1$</th><th>Base 2</th></tr>
<tr><td rowspan="2">101</td><td></td><td>|</td><td>|</td><td></td><td></td><td>|</td><td></td><td>|</td><td rowspan="2">1100101</td></tr>
<tr><td>0</td><td>1</td><td>1</td><td>0</td><td>0</td><td>1</td><td>0</td><td>1</td></tr>
</table>

Example 5

To convert 3850_{10} to base 9, we will fill in the conversion table by writing a tally mark every time we subtract a power.

First, we note that $9^3 = 729$ is the largest power of 9 that is less than 3850 and $3850 - 5(729) = 205$ (so we subtracted 729 five times). Then, $9^2 = 81$ is the largest power of 9 that is less than 205 and $205 - 2(81) = 43$. Then, $9^1 = 9$ is the largest power of 9 that is less than 43 and $43 - 4(9) = 7$. Then, $9^0 = 1$ is the largest power of 9 that is less than 7 and $7 - 7(1) = 0$.

Now, we fill in the bottom row by writing the numeric value of the tallies for each column. The number on the bottom row, read from left to right, is the base 9 number equivalent to the decimal number. So, 3850_{10} in a base 9 system is 5247_9.

Base 10	$9^4 = 6561$	$9^3 = 729$	$9^2 = 81$	$9^1 = 9$	$9^0 = 1$	Base 9
3850		𝍸	||	||||	𝍸||	5247
	0	5	2	4	7	

Binary Number System

The **binary number system** is in base 2 and uses the digits 0 and 1. Since the base is 2, the weights of the positions increase by a factor of 2, unlike the decimal system which increases by 10. So the positions would be named the ones, the twos, the fours, the eights, and so on.

Since binary is in base 2, the digits are used as the symbols, so their values in base 10 are just their numeric values.

Binary Symbols	0	1
Decimal Values	0	1

To convert from binary to decimal or from decimal to binary, use the general methods described in the previous sections.

Example 6

To convert 1011_2 to decimal, we write out the expansion.

$$\begin{aligned} 1011_2 &= (1_2 \times 2^3) + (0_2 \times 2^2) + (1_2 \times 2^1) + (1_2 \times 2^0) \\ &= (1 \times 8) + (0 \times 4) + (1 \times 2) + (1 \times 1) \\ &= 8 + 0 + 2 + 1 \\ &= 11 \end{aligned}$$

Thus, $1011_2 = 11_{10}$.

Exercise 6

Convert the following to decimal.

a) 1_2 b) 10_2 c) 11101_2

Example 7

To convert 45_{10} to binary, we will fill in the conversion table by writing a tally mark every time we subtract a power.

First, we note that $2^5 = 32$ is the largest power of 2 that is less than 45 and $45 - 32 = 13$. Then, $2^3 = 8$ is the largest power of 2 that is less than 13 and $13 - 8 = 5$. Then, $2^2 = 4$ is the largest power of 2 that is less than 5 and $5 - 4 = 1$. Then, 2^0=1 is the largest power of 2 that is less than (or equal to) 1 and $1 - 1 = 0$.

Now, we fill in the bottom row by placing a 1 in every column with one tally and a 0 in every column with no tallies. The number on the bottom row, read from left to right, is the binary number equivalent to the decimal number. So, 45_{10} in binary is 101101_2.

Base 10	$2^6 = 64$	$2^5 = 32$	$2^4 = 16$	$2^3 = 8$	$2^2 = 4$	$2^1 = 2$	$2^0 = 1$	Base 2
45		|		|	|		|	101101
	0	1	0	1	1	0	1	

Exercise 7

Convert the following to binary. Use the conversion table from Example 7.

a) 17_{10} b) 10_{10} c) 0_{10}

Octal Number System

The **octal number system** is in base 8 and uses the digits 0, 1, 2, 3, 4, 5, 6, and 7. Since the base is 8, the weights of the positions increase by a factor of 8.

Since octal is in base 8, the digits are used as the symbols, so their values in base 10 are just their numeric values.

Octal Symbols	0	1	2	3	4	5	6	7
Decimal Values	0	1	2	3	4	5	6	7

To convert from octal to decimal or from decimal to octal, use the general methods described in the previous sections.

Example 8

To convert 237_8 to decimal, we write out the expansion.

$$\begin{aligned}237_8 &= (2_8 \times 8^2) + (3_8 \times 8^1) + (7_8 \times 8^0)\\ &= (2 \times 64) + (3 \times 8) + (7 \times 1)\\ &= 128 + 24 + 7\\ &= 159\end{aligned}$$

Thus, $237_8 = 159_{10}$.

Exercise 8

Convert the following to decimal.

a) 7_8 b) 11_8 c) 11027_8

Example 9

To convert 4293_{10} to octal, we will fill in the conversion table by writing a tally mark every time we subtract a power.

First, we note that $8^4 = 4096$ is the largest power of 8 that is less than 4293 and $4293 - 4096 = 197$. Then, $8^2 = 64$ is the largest power of 8 that is less than 197 and $197 - 64 = 133$. Then, $8^2 = 64$ is the largest power of 8 that is less than 133 and $133 - 64 = 69$. Then, $8^2 = 64$ is the largest power of 8 that is less than 69 and $69 - 64 = 5$. Then, $8^0 = 1$ is the largest power of 8 that is less than 5 and $5 - 5(1) = 0$.

Now, we fill in the bottom row by writing the numeric value of the tallies for each column. The number on the bottom row, read from left to right, is the octal number equivalent to the decimal number. So, 4293_{10} in octal is 10305_8.

<table>
<tr><th>Base 10</th><th>$8^5 = 32768$</th><th>$8^4 = 4096$</th><th>$8^3 = 512$</th><th>$8^2 = 64$</th><th>$8^1 = 8$</th><th>$8^0 = 1$</th><th>Base 8</th></tr>
<tr><td rowspan="2">4293</td><td></td><td>|</td><td></td><td>|||</td><td></td><td>𝍸</td><td rowspan="2">10305</td></tr>
<tr><td>0</td><td>1</td><td>0</td><td>3</td><td>0</td><td>5</td></tr>
</table>

Exercise 9

Convert the following to octal. Use the conversion table from Example 9.

a) 2_{10} b) 8_{10} c) 1089_{10}

Hexadecimal Number System

The **hexadecimal number system** is in base 16. Since the base is 16, the weights of the positions increase by a factor of 16.

The hexadecimal number system is interesting because there are only 10 digits that are used to form numbers in the decimal system, but the hexadecimal system requires 16 symbols. All the positional number systems that we've looked at so far have had bases that are ten or less, so we've just used digits. In hexadecimal, we use the ten digits followed by uppercase letters for the last 6 symbols. In a positional number system, every symbol is assigned a numeric value (in the decimal system) from 0 to one less than the base. For the digits, we've been assigning their own numeric values, so the letters in hexadecimal will be assigned values 10 to 15.

Hexadecimal Symbols	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
Decimal Values	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

To convert from hexadecimal to decimal or from decimal to hexadecimal, use the general methods described in the previous sections.

Example 10

To convert $A03E_{16}$ to decimal, we write out the expansion using the symbols and then substitute the symbols with their values according to the conversion chart.

$$\begin{aligned} A03E_{16} &= (A_{16} \times 16^3) + (0_{16} \times 16^2) + (3_{16} \times 16^1) + (E_{16} \times 16^0) \\ &= (10 \times 16^3) + (0 \times 16^2) + (3 \times 16^1) + (14 \times 16^0) \\ &= (10 \times 4096) + (0 \times 256) + (3 \times 16) + (14 \times 1) \\ &= 40960 + 0 + 48 + 14 \\ &= 41022 \end{aligned}$$

Thus, $A03E_{16} = 41022_{10}$.

Exercise 10

Convert the following to decimal.

a) 9_{16} b) $2F_{16}$ c) $1A2D_{16}$

Example 11

To convert 5058_{10} to hexadecimal, we will fill in the conversion table by writing a tally mark every time we subtract a power.

First, we note that $16^3 = 4096$ is the largest power of 16 that is less than 5058 and $5058 - 4096 = 962$. Then, $16^2 = 256$ is the largest power of 16 that is less than 962 and we can subtract this power three times, so $962 - 3(256) = 194$. Then, $16^1 = 16$ is the largest power of 16 that is less than 194 and we can subtract this power twelve times, so $194 - 12(16) = 2$. Then, $16^0 = 1$ is the largest power of 16 that is less than 2 and $2 - 2(1) = 0$.

Now, we fill in the bottom row by writing the numeric value of the tallies for each column. The number on the bottom row, read from left to right, is the hexadecimal number equivalent to the decimal number. So, 5058_{10} in hexadecimal is $13C2_{16}$.

<table>
<tr><th>Base 10</th><th>$16^3 = 4096$</th><th>$16^2 = 256$</th><th>$16^1 = 16$</th><th>$16^0 = 1$</th><th>Base 16</th></tr>
<tr><td rowspan="2">5058</td><td>|</td><td>|||</td><td>𝍸 𝍸 ||</td><td>||</td><td rowspan="2">$13C2$</td></tr>
<tr><td>1</td><td>3</td><td>C</td><td>2</td></tr>
</table>

Exercise 11

Convert the following to hexadecimal. Use the conversion table from Example 11.

a) 11_{10} b) 17_{10} c) 3740_{10}

Babylonian Number System

The **Babylonian number system** uses a base 60 system of symbols, this is called a **sexagesimal** number system. The Babylonian number system is similar to the decimal system in that they use positions or columns, so that the same digit or symbol in different positions or columns will have different values.

The number 60 is special because 60 can be divided evenly by many whole numbers: 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60.

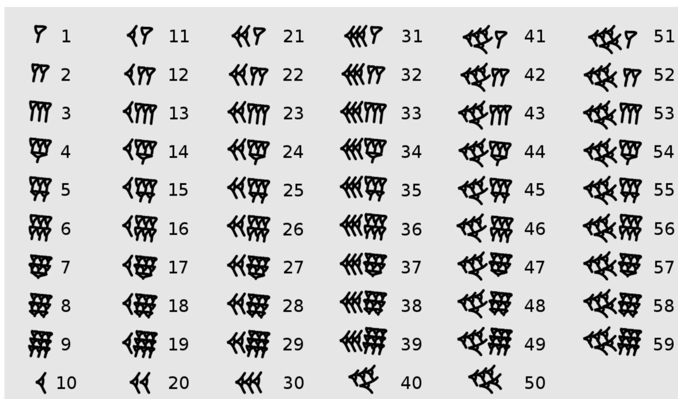


Figure 3: Retrieved from Wikipedia.

Stop and Think

What everyday units of measure use base 60?

Other Number Systems

In this lesson we've explored positional number systems with base 2 (binary), base 8 (octal), base 10 (decimal), base 16 (hexadecimal), and briefly base 60 (Babylonian). However, there are many other possible bases for number systems. It is also not required to use digits and uppercase letters for symbols, the positional number systems would still work with any symbols as long as they are uniquely assigned the decimal number values from 0 to one less than the base.

For conversions into or out of decimal, follow the conversion methods explained for general positional number systems.